On the Convergence of Some Generalized Iterative Methods

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ABSTRACT

This paper is concerned with the generalized accelerated overrelaxation (GAOR) method, which constitutes a generalization of the basic iterative methods for the solution of linear systems. A number of new theoretical results are presented concerning the convergence theory of the GAOR method and special cases of it. Much attention is given to linear systems with positive definite coefficient matrices. Although the problem of determining the various optimum parameters of the GAOR method in the general case is very difficult to solve theoretically and is still an open one, the numerical examples which are presented in this paper show that a suitable exploitation of it may give much better results than the other basic iterative methods.

1. INTRODUCTION AND PRELIMINARIES

In [4] a method which generalizes the basic iterative methods for the solution of linear systems was proposed. According to [4], in order to solve the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n,n}$, $\det(A) \neq 0$, and $x, b \in \mathbb{C}^n$, the following first order iterative method can be used:

$$\begin{aligned} x^{(m+1)} = {} & [D_1 - r(D_2 + C_L)]^{-1} \\ & \times [(1-\omega)D_1 + (\omega - r)(D_2 + C_L) + \omega(D_3 + C_U)]\,x^{(m)} \\ & + \omega[D_1 - r(D_2 + C_L)]^{-1} b, \qquad m = 0, 1, 2, \ldots . \end{aligned} \tag{1.2}$$

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Here D_1, D_2, D_3 are diagonal matrices satisfying

$$D_1 - D_2 - D_3 = D_A \equiv \operatorname{diag}(A), \qquad \det(D_1) \neq 0, \tag{1.3}$$

$-C_L$ and $-C_U$ are the strictly lower and upper triangular parts of A, and $\omega, r \in \mathbb{R}$ $(\omega \neq 0)$, with $\det(D_1 - rD_2) \neq 0$. The method (1.2) is called generalized accelerated overrelaxation(GAOR) method (constituting itself a generalization of the AOR method introduced in [3]) and ω, r are the overrelaxation and acceleration parameters respectively. The iteration matrix of the GAOR method is denoted by $L_{\omega,r}(D_1, D_2) \equiv L_{\omega,r}$ and can be written in the following equivalent forms:

$$L_{\omega,r} = (I - rL)^{-1}[(I - \omega D) + (\omega - r)L + \omega U], \tag{1.4}$$

$$L_{\omega,r} = I - \omega(D_1 - rD_2 - rC_L)^{-1}A, \tag{1.5}$$

where

$$D = (D_1 - rD_2)^{-1}D_A, \qquad L = (D_1 - rD_2)^{-1}C_L, \qquad U = (D_1 - rD_2)^{-1}C_U. \tag{1.6}$$

For specific pairs (ω, r) the GAOR method can give the following simpler generalized methods. Thus:

(i) For $(\omega, r) = (1, 0)$, we have the generalized Jacobi (GJ) method with iteration matrix $L_{1,0}(D_1, D_2) \equiv B = I - D_1^{-1}A = I - D + L + U$, where, because of (1.6),

$$D = D_1^{-1}D_A, \qquad L = D_1^{-1}C_L, \qquad U = D_1^{-1}C_U. \tag{1.7}$$

(ii) For $(\omega, r) = (1, 1)$ we take the generalized Gauss-Seidel (GGS) method with iteration matrix $L_{1,1}(D_1, D_2) \equiv L_{1,1}$.

(iii) For $(\omega, r) = (\omega, \omega)$ we take the generalized SOR (GSOR) method with iteration matrix

$$\begin{aligned} L_{\omega,\omega}(D_1, D_2) \equiv L_{\omega,\omega} &= I - \omega(D_1 - \omega D_2 - \omega C_L)^{-1}A \\ &= (I - \omega L)^{-1}(I - \omega D + \omega U), \end{aligned} \tag{1.8}$$

where D, L, U are given by (1.6). It must also be noted that for $(\omega, r) =$

$(\omega, 0), (\omega, 1)$ we obtain the extrapolated methods of the GJ and GGS respectively, denoted as EGJ and EGGS, with iteration matrices B_ω and $L_{\omega,1}$ respectively. In addition (see [4]), the GAOR method for $r \neq 0$ is an extrapolated GSOR method (EGSOR). It is clear that for $D_1 = D_A$ and $D_2 = 0$ the scheme (1.2), for the previously considered pairs (ω, r), gives the classical iterative methods (see e.g. [17], [20]). Among them the ESOR one was first studied in [14–16] and later in [11], [9], [1] and, almost simultaneously in the form of the AOR method, in [3], [19], [5], [13], [6–8], and [18].

In [4] a study of the GAOR method was made and some first results concerning various cases were obtained when A in (1.1) had certain properties. The purpose of this paper is to present some new theoretical results (supported by numerical ones) concerning the convergence of the GAOR and the related generalized methods. Most of the results generalize and extend the corresponding ones given in [3–4], [12], [19], and [20] for the classical and generalized methods.

2. CONVERGENCE THEORY OF THE GAOR METHOD

In the sequel we shall consider the cases where A in (1.1) is a real symmetric positive definite or an L-matrix (see e.g. [20]) and shall restrict ourselves to $D_1, D_2, D_3 \in \mathbb{R}^n$.

The following Theorem 1 and its Corollaries 1–3 give sufficient conditions for the convergence of the GAOR and the related generalized methods when A is a positive definite matrix.

THEOREM 1. *If A is a real symmetric positive definite matrix, then the GAOR method converges if*

$$M = \omega^{-1}[2D_1 - 2rD_2 - rD_A - (\omega - r)A] \tag{2.1}$$

is positive definite. Moreover $\|L_{\omega,r}\|_{A^{1/2}} < 1$. Conversely, if $\|L_{\omega,r}\|_{A^{1/2}} < 1$ then M is positive definite.

Proof. This theorem is Theorem 3–5.3 of Young [20] when applied to the GAOR method. ∎

COROLLARY 1. *Let A be a positive definite (real, symmetric) matrix. Then the GAOR method converges $[\rho(L_{\omega,r}) < 1]$ if any one of the following* *statements holds*:

(i) $0 < \omega \leqslant r$ *and* $d_{1j} > r\,(d_{2j} + a_{jj}/2)$, $j = 1(1)n$,
(ia) $0 < \omega \leqslant r$, $D_1 > 0$, $D_2 > -D_A/2$, *and* $r \in (0, \min_j \varepsilon_j)$,
(ib) $0 < \omega \leqslant r$, $D_1 < 0$, $D_2 < -D_A/2$, *and* $r \in (\max_j \varepsilon_j, +\infty)$,
(ic) $0 < \omega \leqslant r$, $D_1 > 0$, *and* $D_2 < -D_A/2$,
(ii) $r \leqslant \omega < 0$ *and* $d_{1j} < r(d_{2j} + a_{jj}/2)$, $j = 1(1)n$,
(iia) $r \leqslant \omega < 0$, $D_1 > 0$, $D_2 < -D_A/2$, *and* $r \in (-\infty, \min_j \varepsilon_j)$,
(iib) $r \leqslant \omega < 0$, $D_1 < 0$, $D_2 > -D_A/2$, *and* $r \in (\max_j \varepsilon_j, 0)$,
(iic) $r \leqslant \omega < 0$, $D_1 < 0$, *and* $D_1 < -D_A/2$,

where $\varepsilon_j = 2d_{1j}/(2d_{2j} + a_{jj})$, $j = 1(1)n$, *and where* d_{1j}, d_{2j}, a_{jj}, $j = 1(1)n$, *are the diagonal elements of* D_1, D_2, D_A *respectively*. (*Note*: *For diagonal matrices the inequalities refer to their diagonal elements only*.)

Proof. The proof will be given only for the cases (i), (ia) and is based on the positive definiteness of M in (2.1) above (see also [4], [12], or [2, pp. 190–191]); the others can be studied similarly.

(i): If $0 < \omega \leqslant r$ then the matrix $-(\omega - r)A$ is nonnegative definite. Also, if $d_{1j} > r(d_{2j} + a_{jj}/2)$ for all $j = 1(1)n$, then the matrix $2D_1 - 2rD_2 - rD_A$ is positive definite. Consequently by (2.1) of the theorem, M is positive definite.

(ia): If $0 < \omega \leqslant r$ then the matrix $-(\omega - r)A$ is nonnegative definite. Since $D_1 > 0$ and $2D_2 + D_A > 0$, we shall have $2D_1 - 2rD_2 - rD_A = 2D_1 - r(2D_2 + D_A) > 0$ iff $r < 2d_{1j}/(2d_{2j} + a_{jj})$ for all $j = 1(1)n$, or equivalently iff $r < \min_j \varepsilon_j$, which is valid. From (2.1) now it is clear that M is positive definite. ∎

COROLLARY 2. *If A is a positive definite* (*real*, *symmetric*) *matrix*, *then for $D_2 = 0$ the GAOR method converges* $[\rho(L_{\omega,r}) < 1]$ *if any one of the following statements holds*:

(i) $0 < \omega \leqslant r$ *and* $d_{1j} > ra_{jj}/2$, $j = 1(1)n$,
(ia) $0 < \omega \leqslant r$, $D_1 > 0$, *and* $r \in (0, \min_j \varepsilon_j)$,
(ii) $r \leqslant \omega < 0$ *and* $d_{1j} < ra_{jj}/2$, $j = 1(1)n$,
(iia) $r \leqslant \omega < 0$, $D_1 < 0$, *and* $r \in (\max_j \varepsilon_j, 0)$ *with* $\varepsilon_j = 2d_{1j}/a_{jj}$.

COROLLARY 3. *Let A be a positive definite* (*real*, *symmetric*) *matrix*. *Then*

(i) $\|B\|_{A^{1/2}} < 1$ *if* $2D_1 - A$ *is positive definite*,
(ii) $\|B_\omega\|_{A^{1/2}} < 1$ *if* $2\omega^{-1}D_1 - A$ *is positive definite*,
(iii) $\|L_{1,1}\|_{A^{1/2}} < 1$ *if* $D_3 > -D_A/2$,
(iv) $\|L_{\omega,\omega}\|_{A^{1/2}} < 1$ *if the conditions of Corollary* 1 *are satisfied for* $w = r$.

Proof. For $(\omega, r)=(1,0)$, $(\omega,0)$, $(1,1)$, and (ω,ω) and the specific assumptions in each case, the corresponding matrix M in (2.1) becomes positive definite and the conclusions follow. ■

Lemmas 1 and 2, which are stated and proved below, are very useful in the sequel.

LEMMA 1. *If A is a real symmetric matrix and $D_1 > 0$, then $\mu_j < 1$, $j = 1(1)n$, iff A is positive definite (where μ_j, $j = 1(1)n$ are the eigenvalues of the iteration matrix B of the GJ method).*

Proof. We consider the splitting $A = D_1 - C$, where $D_1 > 0$ and $C = D_2 + D_3 + C_L + C_U$. Then $B = D_1^{-1}C$. We set $\hat{A} = D_1^{-1/2}AD_1^{-1/2} = I - D_1^{-1/2}CD_1^{-1/2} = I - D_1^{1/2}BD_1^{-1/2} = I - \tilde{B}$. If A is positive definite, then $\hat{A} = I - \tilde{B}$ is also positive definite. Therefore the eigenvalues of $I - \tilde{B}$ are $1 - \mu_j > 0$, $j = 1(1)n$ (B and $\tilde{B}$ are similar matrices), that is, $\mu_j < 1$, $j = 1(1)n$. Conversely, if $\mu_j < 1$, $j = 1(1)n$, then the matrix $\hat{A} = I - \tilde{B}$, which is real symmetric, has positive eigenvalues. Consequently $\hat{A}$ is positive definite and so is A. ■

LEMMA 2. *Let A be a real symmetric matrix with $D_A > 0$. Then the matrix $2\omega^{-1}D_1 - A$, where $D_1 > 0$ and $\omega \in \mathbb{R}$ ($\omega \neq 0$), is positive definite if either of the following statements holds*:

$$\mu_j < 1, \quad j = 1(1)n, \quad \text{and} \quad 0 < \omega < \frac{2}{1 - \mu_{\min}}, \tag{2.2}$$

$$\mu_j \geqslant 1, \quad j = 1(1)n, \quad \text{and} \quad \omega \in \left(-\infty, \frac{2}{1 - \mu_{\min}} \right) \cup (0, +\infty), \tag{2.3}$$

where μ_j, $j = 1(1)n$, are the eigenvalues of B and $\mu_{\min} = \min_j \mu_j$.

[Note: If $\mu_j = 1$ for some j in (2.3) then we have only $\omega \in (0, +\infty)$.]

Proof. According to [20, Theorem 2-2.8] the matrix $2\omega^{-1}D_1 - A$ is positive definite iff the matrix $H = D_1^{-1/2}(2\omega^{-1}D_1 - A)D_1^{-1/2}$ is positive definite. We have $H = D_1^{-1/2}(2\omega^{-1}D_1 - A)D_1^{-1/2} = 2\omega^{-1}I - D_1^{-1/2}AD_1^{-1/2} = 2\omega^{-1}I - \hat{A} = 2\omega^{-1}I - (I - \tilde{B}) = (2\omega^{-1} - 1)I + \tilde{B}$, where $\tilde{B} = D_1^{1/2}BD_1^{-1/2} = I - \hat{A}$. The eigenvalues of H are $2\omega^{-1} - 1 + \mu_j$, $j = 1(1)n$, where μ_j, $j = 1(1)n$ are the eigenvalues of B. Since A is real symmetric, the matrices $\hat{A}$, $\tilde{B}$, H are also real symmetric. Therefore μ_j are real, implying that the eigenvalues of H are real. We distinguish the two cases:

(i) $\mu_j < 1,\ j = 1(1)n$. Then the matrix H is positive definite iff

$$2\omega^{-1} - 1 + \mu_j > 0, \qquad j = 1(1)n, \tag{2.4}$$

or equivalently $0 < \omega < 2/(1-\mu_j),\ j = 1(1)n$, that is,

$$0 < \omega < \frac{2}{1-\mu_{\min}}. \tag{2.5}$$

(ii) $\mu_j \geqslant 1,\ j = 1(1)n$. If $\omega > 0$ then (2.4) is always valid. If $\omega < 0$ then we must have from (2.4) that $\omega < 2/(1-\mu_j),\ j = 1(1)n,\ \mu_j \neq 1$. Therefore $\omega \in (-\infty, 2/(1-\mu_{\min})) \cup (0, +\infty)$. ∎

The following theorem gives sufficient and necessary conditions for the convergence of the EGJ method when A is positive definite.

THEOREM 2. *Let A be a positive definite matrix. Then the EGJ method with $D_1 > 0$ converges iff $2\omega^{-1}D_1 - A$ is positive definite or equivalently iff $0 < \omega < 2/(1-\mu_{\min})$, where $\mu_{\min}$ is the minimum eigenvalue of the iteration matrix B of the GJ method.*

Proof. The first part of the theorem arises from case (ii) of Corollary 2. Suppose now that $\rho(B_\omega) < 1$. According to Lemma 1 the eigenvalues of the matrix B satisfy $\mu_j < 1,\ j = 1(1)n$. Since

$$B_\omega = \omega B + (1-\omega)I, \tag{2.6}$$

we have $\lambda_j = \omega\mu_j + 1 - \omega$, where λ_j are the eigenvalues of B_ω and $|\lambda_j| < 1$, $j = 1(1)n$. Now $|\omega\mu_j + 1 - \omega| < 1$ is equivalent to

$$\omega(\mu_j - 1) < 0 \quad \text{and} \quad -2 < \omega(\mu_j - 1). \tag{2.7}$$

From (2.7) it is implied that $0 < \omega < 2/(1-\mu_j),\ j = 1(1)n$, that is, $0 < \omega < 2/(1-\mu_{\min})$. By Lemma 2 now we conclude that $2\omega^{-1}D_1 - A$ is positive definite. ∎

COROLLARY 4. *Let A be a real symmetric matrix with $D_A > 0$. Then the GJ method with $D_1 > 0$ converges $[\rho(B) < 1]$ iff A and $2D_1 - A$ are positive definite.* ∎

Lemmas 3 and 4 below are useful in the proof of the Theorem 3, which is concerned with sufficient and necessary conditions for a class of GAOR methods to converge.

LEMMA 3. *If* $D_1 = kD_A$, $k \in \mathbb{R}$ $(k \neq 0)$, *then*

$$B = \left(1 - \frac{1}{k}\right)I + \frac{1}{k}B',$$

where B, B' are the iteration matrices of the GJ and J methods respectively. That is, the GJ method is an extrapolated method of the Jacobi (J) method with extrapolation parameter $1/k$.

Proof. $B = I - D + L + U$ with $D = (kD_A)^{-1}D_A = (1/k)I$, $L = (kD_A)^{-1}C_L = (1/k)L'$, $U = (kD_A)^{-1}C_u = (1/k)U'$, and $B' = L' + U'$. Thus

$$B = I - \frac{1}{k}I + \frac{1}{k}(L' + U') = \left(1 - \frac{1}{k}\right)I + \frac{1}{k}B'. \qquad \blacksquare$$

LEMMA 4. *Let A be a real symmetric matrix with $D_A > 0$. If $D_1 = kD_A$ $(k > 0)$, $D_2 = 0$, $D_3 \geqslant 0$, then the matrix $M = \omega^{-1}[(2k - r)D_A + (r - \omega)A]$ $(\omega \neq 0)$ is positive definite iff*

$$0 < \omega < 2k \quad \text{and} \quad \omega + \frac{2k - \omega}{\mu'_{\min}} < r < \omega + \frac{2k - \omega}{\mu'_{\max}}, \tag{2.8}$$

where $\mu'_{\min}$, $\mu'_{\max}$ are the minimum and the maximum eigenvalues of the matrix B' of the J method.

Proof. First we observe that $D_1 = kD_A = D_A + D_3 > 0$. The matrix M is positive definite iff the following matrix is positive definite:

$$H = D_A^{-1/2}M\left(D_A^{-1/2}\right)^H = D_A^{-1/2}MD_A^{-1/2} = \omega^{-1}(2k - r)I + \omega^{-1}(r - \omega)\hat{A},$$

where $\hat{A} = D_A^{-1/2}AD_A^{-1/2} = k(I - \tilde{B})$, $\tilde{B} = D_A^{1/2}BD_A^{-1/2}$, and B is the iteration matrix of the GJ method. Hence

$$\begin{aligned} H &= \omega^{-1}(2k - r)I + \omega^{-1}(r - \omega)k(I - \tilde{B}) \\ &= \omega^{-1}\left[2k - r + k(r - \omega)\right]I + \omega^{-1}k(\omega - r)\tilde{B}. \end{aligned}$$

Evidently H is real symmetric with eigenvalues

$$\lambda_j = \omega^{-1}[2k - r + k(r-\omega)] + \omega^{-1}k(\omega - r)\mu_j, \qquad j = 1(1)n, \tag{2.9}$$

where μ_j are the eigenvalues of B. According to Lemma 3 we have

$$\mu_j = 1 - \frac{1}{k} + \frac{\mu'_j}{k}, \qquad j = 1(1)n, \tag{2.10}$$

where μ'_j, $j = 1(1)n$, are the eigenvalues of the iteration matrix B' of the J method. By (2.9) and (2.10) we obtain

$$\lambda_j = \omega^{-1}(2k - \omega) + \omega^{-1}(\omega - r)\mu'_j. \tag{2.11}$$

Since $\operatorname{tr}(B') = 0$ and μ'_j are real, we must have $\mu'_{\min} \leqslant 0$ and $\mu'_{\max} \geqslant 0$ with $\mu'_{\min} \leqslant \mu'_j \leqslant \mu'_{\max}$. The matrix H is positive definite iff $\lambda_j > 0$, $j = 1(1)n$. In the sequel we assume that $\mu'_{\min} \neq \mu'_{\max}$. From (2.11) we obtain

$$\omega^{-1}(2k - \omega) > \omega^{-1}(\omega - r)\mu'_{\min} \quad \text{and} \quad \omega^{-1}(2k-\omega) > \omega^{-1}(\omega - r)\mu'_{\max}. \tag{2.12}$$

Since the product of the right hand sides of the inequalities (2.12) is zero for $\omega = r$ and negative for $\omega \neq r$ (because $\mu'_{\min}\mu'_{\max} < 0$), at least one of the factors will be nonnegative, making the corresponding left hand sides of (2.12) positive. In either case it is derived that $0 < \omega < 2k$ (since $k > 0$). Solving now each one of (2.12) for r, the second series of inequalities in (2.8) is obtained and the proof is completed. ∎

REMARKS.

(i) If $\mu'_{\min} = \mu'_{\max} = 0$, then we easily find that $\lambda_j > 0$ iff $0 < \omega < 2k$.
(ii) If $r = 0$, then the second of (2.8) gives $0 < \omega < 2k/(1 - \mu'_{\min}) \leqslant 2k$. ∎

THEOREM 3. *Let A be a real symmetric matrix with $D_A > 0$ and $D_1 = kD_A$ $(k > 0)$, $D_2 = 0$, $D_3 \geqslant 0$. If* (2.8) *hold, then the* GAOR *method converges* $[\rho(L_{\omega,r}) < 1]$ *iff A is positive definite.*

Proof. Following Theorem 1, it is sufficient to show that under the validity of (2.8) the matrix M, given by (2.1) with $D_2 = 0$ and $D_1 = kD_A$, is positive definite:

$$M = \omega^{-1}\left[(2k - r)D_A - (\omega - r)A\right].$$

However, Lemma 4 guarantees the positive definiteness of M. ■

COROLLARY 5. *Let A be a real symmetric matrix with $D_A > 0$. If $D_1 = kD_A$ $(k > 0)$, then the matrix $2\omega^{-1}D_1 - A$ is positive definite iff $0 < \omega < 2k/(1 - \mu'_{\min})$, where $\mu'_{\min} = \min_j \mu'_j$ and μ'_j, $j = 1(1)n$, are the eigenvalues of the iteration matrix B' of the J method.*

Proof. For $r = 0$ the matrix M in Theorem 1 takes the form $M = \omega^{-1}[2kD_A - \omega A] = 2\omega^{-1}D_1 - A$. Now, according to remark (ii) following Lemma 4, the proof is obvious. ■

The following theorems are concerned with the case where the matrix A in (1.1) is an L-matrix.

THEOREM 4. *Let A be an L-matrix, $D_2, D_3 \geqslant 0$, and $0 \leqslant r \leqslant 1$. Then A is an M-matrix if $\rho(F) < 1$, where $F = I - (D_1 - rD_2)^{-1}A$.*

Proof. If $\rho(F) < 1$, then $I - F$ is nonsingular and the series $I + F + F^2 + \cdots$ converges to $(I - F)^{-1}$. Since $D_1 - rD_2 \geqslant D_1 - D_2 \geqslant D_1 - D_2 - D_3 = D_A > 0$, we obtain $D_1 - rD_2 > 0$. Setting $A = D_1 - C$, where $C = D_2 + D_3 + C_L + C_U$, it is implied that $C \geqslant 0$. Thus we have $F = I - (D_1 - rD_2)^{-1}A = (D_1 - rD_2)^{-1}(-rD_2 + C) = (D_1 - rD_2)^{-1}((1 - r)D_2 + D_3 + C_L + C_U) \geqslant 0$, implying that $(I - F)^{-1} \geqslant 0$. Since $D_1 - rD_2$ and $I - F$ are nonsingular matrices, we obtain that $A = (D_1 - rD_2)(I - F)$ is nonsingular and $A^{-1} = (I - F)^{-1}(D_1 - rD_2)^{-1} \geqslant 0$, that is, A is an M-matrix. ■

THEOREM 5. *If A is an L-matrix and $D_1 \geqslant D_A$, then A is an M-matrix iff the GJ method converges $[\rho(B) < 1]$.*

Proof. The proof is analogous to that of Theorem 2-7.2 in [20], where we note that if $A = D_1 - C$, with $C = D_2 + D_3 + C_L + C_U$, then $C = D_1 - D_A + C_L + C_U \geqslant 0$. ■

3. APPLICATIONS AND NUMERICAL EXAMPLES

In this section we present three categories of applications and examples. The first deals theoretically with the optimum GSOR method in the trivial case of $A \in \mathbb{R}^{2,2}$. In the second one three numerical examples with $A \in \mathbb{R}^{3,3}$ are treated computationally, and the superiority of the best GSOR method obtained in this way over the classical SOR and AOR ones is shown. In the third application we consider the Laplace equation in a square approximated by a 9-point stencil, and to the resulting linear system the GAOR method is applied. It is numerically proved that the method in question is faster than the classical AOR one.

A.

We consider first the very special case where the matrix A of the system (1.1) is a real matrix of order two. In [12] the convergence of the SOR method is examined for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad ad \neq 0 \text{ and } \det(A) \neq 0.$$

As we can see from the results in [12], we have $\rho(L_{\omega,\omega}) = 0$ iff $\omega = 1$ and $bc = 0$, that is, only in the case of the Gauss-Seidel method on a triangular matrix. Suppose now that we have the real matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{where} \quad |a| + |d| \neq 0 \text{ and } \det(A) \neq 0,$$

and apply the GSOR method with

$$D_2 = 0, \qquad D_1 = \begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}, \quad \text{and} \quad a_1 d_1 \neq 0.$$

It can be shown that the iteration matrix of the GSOR is

$$L_{\omega,\omega} = \begin{bmatrix} 1 - \dfrac{\omega a}{a_1} & -\dfrac{\omega b}{a_1} \\[2ex] \omega c \dfrac{\omega a - a_1}{a_1 d_1} & \dfrac{\omega^2 cd/a_1 + d_1 - \omega d}{d_1} \end{bmatrix}.$$

The characteristic equation for $L_{\omega,\omega}$ is

$$\lambda^2 - \left(2 - \frac{\omega a}{a_1} - \frac{\omega d}{d_1} + \frac{\omega^2 bc}{a_1 d_1}\right)\lambda + \frac{(a_1 - \omega a)(d_1 - \omega d)}{a_1 d_1} = 0. \quad (3.1)$$

In order to have $\rho(L_{\omega,\omega}) = 0$ we require the two roots of (3.1) to be equal to zero, that is,

$$2 - \frac{\omega a}{a_1} - \frac{\omega d}{d_1} + \frac{\omega^2 bc}{a_1 d_1} = 0 \quad \text{and} \quad \frac{(a_1 - \omega a)(d_1 - \omega d)}{a_1 d_1} = 0. \quad (3.2)$$

From (3.2) we can determine a_1, d_1, and ω, one of which can be chosen arbitrarily. Thus if d_1 is arbitrary ($\neq 0$), then

$$\omega = \frac{ad_1}{ad - bc} \quad \text{and} \quad a_1 = \frac{a^2 d_1}{ad - bc}. \quad (3.3)$$

If a_1 is arbitrary ($\neq 0$), then

$$\omega = \frac{a_1 d}{ad - bc} \quad \text{and} \quad d_1 = \frac{d^2 a_1}{ad - bc}. \quad (3.4)$$

Therefore we can always find a matrix D_1 and a value for ω such that $\rho(L_{\omega,\omega}) = 0$, even in the case where at most one of a, d is zero, a fact which makes the application of the classical SOR method impossible.

REMARK 1. From (3.3) [(3.4)] it is observed that if $a \neq 0$ [$d \neq 0$], then for $\omega = 1$ it is obtained that $a_1 = a$, $d_1 = (ad - bc)/a$ [$a_1 = (ad - bc)/d$, $d_1 = d$], implying $\rho(L_{1,1}) = 0$. In other words, the GGS method is the best and the simplest one to use.

REMARK 2. In the trivial case we have examined, the optimum GSOR (GAOR), which gives $\rho(L_{\omega,\omega}) = 0$ (something one would expect), has been obtained theoretically with the values of the parameters involved being real.

B.

In this subsection three simple numerical examples are presented for which the values of the optimum spectral radius together with those of the optimum parameters involved for the classical SOR and AOR methods were found numerically in [19]. The basic matrix A is a 3×3 positive definite matrix in the first two examples, and an L-matrix in the third one. In all three cases the GSOR method with $D_2=0$ was considered, and this was in order to show that even this rather simple generalized method beats substantially both classical SOR and AOR ones as regards their convergence rates.

(i) If

$$A=\begin{bmatrix}1 & 0.5 & 0.5\\ 0.5 & 1 & 0.5\\ 0.5 & 0.5 & 1\end{bmatrix},$$

then with

$$D_1=\begin{bmatrix}0.5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad \text{and} \quad \omega=1$$

we find that $\rho(L_{\omega,\omega})=0$. We note that, as is shown in [19], for the optimum spectral radius of the SOR method we have $\rho(\text{SOR})\approx 0.330$ with $\omega_{\text{opt}}=1.08$, while for the optimum AOR method we have $\rho(\text{AOR})\approx 0.250$ with $\omega_{\text{opt}}=1.25$, $r_{\text{opt}}=1.00$.

(ii) If

$$A=\begin{bmatrix}1 & 0.4 & 0.4\\ 0.4 & 1 & 0.6\\ 0.4 & 0.6 & 1\end{bmatrix},$$

then with

$$D_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1.6 & 0\\ 0 & 0 & 1.3\end{bmatrix} \quad \text{and} \quad \omega=1.60$$

we find that $\rho(L_{\omega,\omega})\approx 0.051$, while the corresponding optimum results for the SOR and AOR methods are $\rho(\text{SOR})\approx 0.282$ for $\omega_{\text{opt}}=1.80$, and $\rho(\text{AOR})\approx 0.196$ for $\omega_{\text{opt}}=1.23$, $r_{\text{opt}}=1.03$.

(iii) If

$$A=\begin{bmatrix}4 & -1 & 0\\ -1 & 4 & -1\\ 0 & -1 & 4\end{bmatrix},$$

then with

$$D_1 = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{and} \quad \omega = 2$$

it can be shown that $\rho(L_{\omega,\omega}) = 0$. In this case it is found that $\rho(\text{SOR}) = \rho(\text{AOR}) = 0.100$ for $\omega_{\text{opt}} = r_{\text{opt}} = 1.10$.

NOTE. In the examples above D_1 was found computationally by varying its elements over various ranges covering the diagonal elements of the corresponding matrices A.

C.

We consider the numerical solution of the Dirichlet problem for the Laplace equation in the unit square:

$$-\Delta u \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0, \qquad (x,y) \in R,$$

$$u(x,y) = g(x,y), \qquad (x,y) \in \partial R,$$

where $R \cup \partial R = \{(x,y) : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$ and g is a known function. For the numerical solution of the above problem we use the well-known 9-point finite-difference approximation formula at the mesh points $(x_j, x_k) = (jh, kh)$, $j, k = 1(1)N-1$, and $N = 1/h$. For $N = 5$ the resulting linear system has the matrix form

$$Ax = b, \tag{3.5}$$

where

$$A = \begin{bmatrix} B & C & 0 & 0 \\ C & B & C & 0 \\ 0 & C & B & C \\ 0 & 0 & C & B \end{bmatrix}$$

with B and C the 4×4 matrices

$$B = \begin{bmatrix} 20 & -4 & 0 & 0 \\ -4 & 20 & -4 & 0 \\ 0 & -4 & 20 & -4 \\ 0 & 0 & -4 & 20 \end{bmatrix} \quad \text{and} \quad C = -\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

First the AOR method was applied to the system (3.3). The parameters ω and r were given the values 0.01(0.01)2.00 (see e.g. [10, p. 113–114], and we found that the optimum spectral radius $\rho(\text{AOR}) \approx 0.25449$ for $\omega_{\text{opt}} = 1.20$, $r_{\text{opt}} = 1.26$. Then we considered the GAOR method with

$$D_2 = 0 \quad \text{and} \quad D_1 = \begin{bmatrix} E & & & \\ & E & & \\ & & E & \\ & & & E \end{bmatrix},$$

where the diagonal elements of each diagonal block of D_1 were given the values 10(1)35 as follows. Let e_1, e_2, e_3, e_4 be the diagonal elements of E. Initially $e_2^{(0)} = e_2$, $e_3^{(0)} = e_3$, $e_4^{(0)} = e_4$ were kept fixed and $e_1^{(0)} = e_1$ took the values 10(1)35, with varying $\omega, r = 0.01(0.01)2.00$. Let $e_1^{(1)}$ be the value of $e_1^{(0)}$ corresponding to the smallest spectral radius of the GAOR method. Then $e_1^{(1)}, e_3^{(0)}, e_4^{(0)}$ were kept fixed, and $e_2^{(0)}$ was varied in the same way. Let $e_2^{(1)}$ be the best value of $e_2^{(0)}$ in the previous sense. This process went on until $e_4^{(1)}$ was obtained. The whole procedure was applied again in a cyclic way until $e_j^{(2)}$, $j = 1(1)4$, were obtained, and so on. As soon as the set of the four values $e_j^{(k)}$, $j = 1(1)4$, coincided with $e_j^{(k+1)}$, $j = 1(1)4$, the procedure was terminated and the last available values were considered as the best ones. Thus, for

$$E = \begin{bmatrix} 27 & & & 0 \\ & 26 & & \\ & & 24 & \\ 0 & & & 32 \end{bmatrix}$$

the optimum spectral radius was found to be $\rho(L_{\omega,r}) \approx 0.22680$ for $\omega_{\text{opt}} = 1.53$, $r_{\text{opt}} = 1.71$.

NOTE 1. Despite the fact that the optimum spectral radii of the AOR and GAOR methods in the example we presented do not differ very much, the elements of E [although they were varied by using a large step (1)] as well as the values of ω_{opt} and r_{opt} for the GAOR method are quite different from the corresponding ones for the AOR method. This is indicative and strongly suggests that if all the elements of all four blocks E were allowed to vary independently and a small step were used, then the optimum spectral radius for the GAOR method would be much better than the one for the AOR method.

NOTE 2. The numerical procedure which was adopted in the example above and gives a best GAOR method requires much computational effort and is only recommended when the system (3.5) is to be solved with many right hand sides b.

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